7-1 Operations on Functions

Main Ideas

- Find the sum, difference, product, and quotient of functions.
- Find the composition of functions.

New Vocabulary

composition of functions

GET READY for the Lesson

Carol Coffmon owns a store where she sells birdhouses. The revenue from birdhouse sales is given by $r(x) = 125x$. The cost of making the birdhouses is given by $c(x) = 65x + 5400$. Her profit p is the revenue minus the cost or $p = r - c$. So the profit function $p(x)$ can be defined as $p(x) = (r - c)(x)$.

Arithmetic Operations Let $f(x)$ and $g(x)$ be any two functions. You can add, subtract, multiply, and divide functions according to these rules.

KEY CONCEPT *Operations with Functions*

Operation	Definition	Examples if $f(x) = x + 2$, $g(x) = 3x$
Sum	$(f + g)(x) = f(x) + g(x)$	$(x + 2) + 3x = 4x + 2$
Difference	$(f - g)(x) = f(x) - g(x)$	$(x + 2) - 3x = -2x + 2$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(x + 2)3x = 3x^2 + 6x$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$\frac{x + 2}{3x}, x \neq 0$

EXAMPLE **Add and Subtract Functions**

1 Given $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$, find each function.

a. $(f + g)(x)$

$(f + g)(x) = f(x) + g(x)$ Addition of functions

$= (x^2 - 3x + 1) + (4x + 5)$ $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$

$= x^2 + x + 6$ Simplify.

b. $(f - g)(x)$

$(f - g)(x) = f(x) - g(x)$ Subtraction of functions

$= (x^2 - 3x + 1) - (4x + 5)$ $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$

$= x^2 - 7x - 4$ Simplify.

CHECK Your Progress

Given $f(x) = x^2 + 5x - 2$ and $g(x) = 3x - 2$, find each function.

1A. $(f + g)(x)$ **1B.** $(f - g)(x)$

Notice that the functions f and g have the same domain of all real numbers. The functions $f + g$ and $f - g$ also have domains that include all real numbers. For each new function, the domain consists of the intersection of the domains of $f(x)$ and $g(x)$. The domain of the quotient function is further restricted by excluded values that make the denominator equal to zero.

EXAMPLE 2 Multiply and Divide Functions

Given $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$, find each function.

a. $(f \cdot g)(x)$

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) && \text{Product of functions}\\ &= (x^2 + 5x - 1)(3x - 2) && \text{Substitute.}\\ &= x^2(3x - 2) + 5x(3x - 2) - 1(3x - 2) && \text{Distributive Property}\\ &= 3x^3 - 2x^2 + 15x^2 - 10x - 3x + 2 && \text{Distributive Property}\\ &= 3x^3 + 13x^2 - 13x + 2 && \text{Simplify.}\end{aligned}$$

b. $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Division of functions}$$

$$= \frac{x^2 + 5x - 1}{3x - 2}, x \neq \frac{2}{3} \quad f(x) = x^2 + 5x - 1 \text{ and } g(x) = 3x - 2$$

Because $x = \frac{2}{3}$ makes $3x - 2 = 0$, $\frac{2}{3}$ is excluded from the domain of $\left(\frac{f}{g}\right)(x)$.

CHECK Your Progress

Given $f(x) = x^2 - 7x + 2$ and $g(x) = x + 4$, find each function.

2A. $(f \cdot g)(x)$ **2B.** $\left(\frac{f}{g}\right)(x)$

Cross-Curricular Project

Math nline You can use operations on functions to find a function to compare the populations of different cities, states, or countries over time. Visit algebra2.com.

Composition of Functions

Functions can also be combined using **composition of functions**. In a composition, a function is performed, and then a second function is performed on the result of the first function.

KEY CONCEPT *Composition of Functions*

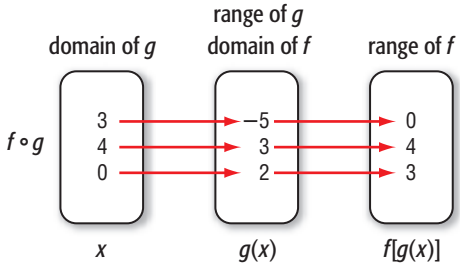
Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composite function $f \circ g$ can be described by

$$[f \circ g](x) = f[g(x)].$$

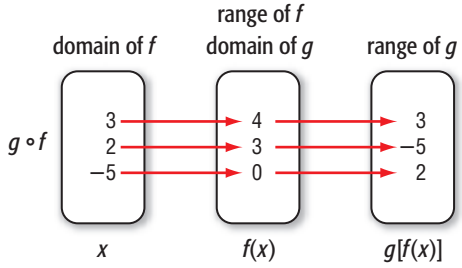
Reading Math

Composite Functions The composition of f and g is denoted by $f \circ g$. This is read *f of g*.

Suppose $f = \{(3, 4), (2, 3), (-5, 0)\}$ and $g = \{(3, -5), (4, 3), (0, 2)\}$.



$f \circ g = \{(3, 0), (4, 4), (0, 3)\}$



$g \circ f = \{(3, 3), (2, -5), (-5, 2)\}$

Concepts in Motion

Animation algebra2.com

The composition of two functions may not exist. Given two functions f and g, $[f \circ g](x)$ is defined only if the range of $g(x)$ is a subset of the domain of $f(x)$.

EXAMPLE 3 Evaluate Composition of Functions

If $f = \{(7, 8), (5, 3), (9, 8), (11, 4)\}$ and $g = \{(5, 7), (3, 5), (7, 9), (9, 11)\}$, find $f \circ g$ and $g \circ f$.

To find $f \circ g$, evaluate $g(x)$ first. Then use the range of g as the domain of f and evaluate $f(x)$.

$f[g(5)] = f(7)$ or 8 $\quad g(5) = 7$ $\qquad$ $f[g(7)] = f(9)$ or 8 $\quad g(7) = 9$

$f[g(3)] = f(5)$ or 3 $\quad g(3) = 5$ $\qquad$ $f[g(9)] = f(11)$ or 4 $\quad g(9) = 11$

$f \circ g = \{(5, 8), (3, 3), (7, 8), (9, 4)\}$

To find $g \circ f$, evaluate $f(x)$ first. Then use the range of f as the domain of g and evaluate $g(x)$.

$g[f(7)] = g(8)$ $\quad g(8)$ is undefined. $\qquad$ $g[f(9)] = g(8)$ $\quad g(8)$ is undefined.

$g[f(5)] = g(3)$ or 5 $\quad f(5) = 3$ $\qquad$ $g[f(11)] = g(4)$ $\quad g(4)$ is undefined.

Since 8 and 4 are not in the domain of g, $g \circ f$ is undefined for $x = 7$, $x = 9$, and $x = 11$. However, $g[f(5)] = 5$ so $g \circ f = \{(5, 5)\}$.

CHECK Your Progress

3. If $f = \{(3, -2), (-1, -5), (4, 7), (10, 8)\}$ and $g = \{(4, 3), (2, -1), (9, 4), (3, 10)\}$, find $f \circ g$ and $g \circ f$.

Notice that in most instances $f \circ g \neq g \circ f$. Therefore, the order in which you compose two functions is very important.

EXAMPLE 4 Simplify Composition of Functions

a. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x + 3$ and $g(x) = x^2 + x - 1$.

$$\begin{aligned}
[f \circ g](x) &= f[g(x)] && \text{Composition of functions} \\
&= f(x^2 + x - 1) && \text{Replace } g(x) \text{ with } x^2 + x - 1. \\
&= (x^2 + x - 1) + 3 && \text{Substitute } x^2 + x - 1 \text{ for } x \text{ in } f(x). \\
&= x^2 + x + 2 && \text{Simplify.}
\end{aligned}$$

$$\begin{aligned}
[g \circ f](x) &= g[f(x)] && \text{Composition of functions} \\
&= g(x + 3) && \text{Replace } f(x) \text{ with } x + 3. \\
&= (x + 3)^2 + (x + 3) - 1 && \text{Substitute } x + 3 \text{ for } x \text{ in } g(x). \\
&= x^2 + 6x + 9 + x + 3 - 1 && \text{Evaluate } (x + 3)^2. \\
&= x^2 + 7x + 11 && \text{Simplify.}
\end{aligned}$$

So, $[f \circ g](x) = x^2 + x + 2$ and $[g \circ f](x) = x^2 + 7x + 11$.

Study Tip

Composing Functions

To remember the correct order for composing functions, think of starting with x and working outward from the grouping symbols.

b. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = 2$.

$[f \circ g](x) = x^2 + x + 2$ Function from part a

$[f \circ g](2) = (2)^2 + 2 + 2$ or 8 Replace x with 2 and simplify.

$[g \circ f](x) = x^2 + 7x + 11$ Function from part a

$[g \circ f](2) = (2)^2 + 7(2) + 11$ or 29 Replace x with 2 and simplify.

So, $[f \circ g](2) = 8$ and $[g \circ f](2) = 29$.

CHECK Your Progress

4A. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x - 5$ and $g(x) = x^2 + 2x + 3$.

4B. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = -3$.

Real-World EXAMPLE **Use Composition of Functions**

Study Tip

Combining Functions

By combining functions, you can make the evaluation of the functions more efficient.

5 **TAXES** **Tyrone Davis has $180 deducted from every paycheck for retirement. He can have these deductions taken before taxes are applied, which reduces his taxable income. His federal income tax rate is 18%. If Tyrone earns $2200 every pay period, find the difference in his net income if he has the retirement deduction taken before taxes or after taxes.**

Explore Let x = Tyrone's income per paycheck, $r(x)$ = his income after the deduction for retirement, and $t(x)$ = his income after the deduction for federal income tax.

Plan Write equations for $r(x)$ and $t(x)$.
$180 is deducted from every paycheck for retirement:
$r(x) = x - 180$.

Tyrone's tax rate is 18%: $t(x) = x - 0.18x$.

Solve If Tyrone has his retirement deducted *before* taxes, then his net income is represented by $[t \circ r](2200)$.

$[t \circ r](2200) = t(2200 - 180)$ Replace x with 2200 in $r(x) = x - 180$.

$= t(2020)$

$= 2020 - 0.18(2020)$ Replace x with 2020 in $t(x) = x - 0.18x$.

$= 1656.40$

If Tyrone has his retirement deducted *after* taxes, then his net income is represented by $[r \circ t](2200)$.

$[r \circ t](2200) = r[2200 - 0.18(2200)]$ Replace x with 2200 in $t(x) \circ x - 0.18x$.

$= r(1804)$

$= 1804 - 180$ Replace x with 1804 in $r(x) = x - 180$.

$= 1624$

$[t \circ r](2200) = 1656.40$ and $[r \circ t](2200) = 1624$. The difference is $1656.40 - $1624, or $32.40. So, his net pay is $32.40 more by having his retirement deducted before taxes.

Check The answer makes sense. Since the taxes are being applied to a smaller amount, less tax will be deducted from his paycheck.

CHECK Your Progress

5. All-Mart is offering both an in-store $35 rebate and a 15% discount on an MP3 player that normally costs $300. Which provides the better price: taking the discount before the rebate, or taking the rebate before the discount?

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CHECK Your Understanding

Examples 1, 2 (pp. 384–385)

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

1. $f(x) = 3x + 4$, $g(x) = 5 + x$

2. $f(x) = x^2 + 3$, $g(x) = x - 4$

Example 3 (p. 386)

For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist.

3. $f = \{(-1, 9), (4, 7)\}$, $g = \{(-5, 4), (7, 12), (4, -1)\}$

4. $f = \{(0, -7), (1, 2), (2, -1)\}$, $g = \{(-1, 10), (2, 0)\}$

Example 4 (pp. 386–387)

Find $[g \circ h](x)$ and $[h \circ g](x)$.

5. $g(x) = 2x$, $h(x) = 3x - 4$

6. $g(x) = x + 5$, $h(x) = x^2 + 6$

If $f(x) = 3x$, $g(x) = x + 7$, and $h(x) = x^2$, find each value.

7. $f[g(3)]$ **8.** $g[h(-2)]$ **9.** $h[h(1)]$

Example 5 (p. 387)

SHOPPING For Exercises 10–13, use the following information.
Mai-Lin is shopping for computer software. She finds a CD-ROM that costs $49.99, but is on sale at a 25% discount. She also has a $5 coupon she can use.

10. Express the price of the CD after the discount and the price of the CD after the coupon. Let x represent the price of the CD, $p(x)$ represent the price after the 25% discount, and $c(x)$ represent the price after the coupon.

11. Find $c[p(x)]$ and explain what this value represents.

12. Find $p[c(x)]$ and explain what this value represents.

13. Which method results in the lower sale price? Explain your reasoning.

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–21	1, 2
22–27	3
28–45	4
46, 47	5

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

14. $f(x) = x + 9$, $g(x) = x - 9$

15. $f(x) = 2x - 3$, $g(x) = 4x + 9$

16. $f(x) = 2x^2$, $g(x) = 8 - x$

17. $f(x) = x^2 + 6x + 9$, $g(x) = 2x + 6$

18. $f(x) = x^2 - 1$, $g(x) = \frac{x}{x + 1}$

19. $f(x) = x^2 - x - 6$, $g(x) = \frac{x - 3}{x + 2}$

WALKING For Exercises 20 and 21, use the following information.
Carlos is walking on a moving walkway. His speed is given by the function $C(x) = 3x^2 + 3x - 4$, and the speed of the walkway is $W(x) = x^2 - 4x + 7$.

20. What is his total speed as he walks along the moving walkway?

21. Carlos turned around because he left his cell phone at a restaurant. What was his speed as he walked against the moving walkway?

For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist.

22. $f = \{(1, 1), (0, -3)\}$
$g = \{(1, 0), (-3, 1), (2, 1)\}$

23. $f = \{(1, 2), (3, 4), (5, 4)\}$
$g = \{(2, 5), (4, 3)\}$

24. $f = \{(3, 8), (4, 0), (6, 3), (7, -1)\}$
$g = \{(0, 4), (8, 6), (3, 6), (-1, 8)\}$

25. $f = \{(4, 5), (6, 5), (8, 12), (10, 12)\}$
$g = \{4, 6), (2, 4), (6, 8), (8, 10)\}$

26. $f = \{(2, 5), (3, 9), (-4, 1)\}$
$g = \{(5, -4), (8, 3), (2, -2)\}$

27. $f = \{(7, 0), (-5, 3), (8, 3), (-9, 2)\}$
$g = \{(2, -5), (1, 0), (2, -9), (3, 6)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

28. $g(x) = 4x$
$h(x) = 2x - 1$

29. $g(x) = -5x$
$h(x) = -3x + 1$

30. $g(x) = x + 2$
$h(x) = x^2$

31. $g(x) = x - 4$
$h(x) = 3x^2$

32. $g(x) = 2x$
$h(x) = x^3 + x^2 + x + 1$

33. $g(x) = x + 1$
$h(x) = 2x^2 - 5x + 8$

If $f(x) = 4x$, $g(x) = 2x - 1$, and $h(x) = x^2 + 1$, find each value.

34. $f[g(-1)]$

35. $h[g(4)]$

36. $g[f(5)]$

37. $f[h(-4)]$

38. $g[g(7)]$

39. $f[f(-3)]$

40. $h\left[f\left(\frac{1}{4}\right)\right]$

41. $g\left[h\left(-\frac{1}{2}\right)\right]$

42. $[g \circ (f \circ h)](3)$

43. $[f \circ (h \circ g)](3)$

44. $[h \circ (g \circ f)](2)$

45. $[f \circ (g \circ h)](2)$

Real-World Link
In 2003, there were an estimated 19.2 million people who participated in inline skating.

Source: Inline Skating Resource Center

POPULATION GROWTH **For Exercises 46 and 47, use the following information.**
From 1990 to 2002, the number of births $b(x)$ in the United States can be modeled by the function $b(x) = -8x + 4045$, and the number of deaths $d(x)$ can be modeled by the function $d(x) = 24x + 2160$, where x is the number of years since 1990 and $b(x)$ and $d(x)$ are in thousands.

46. The net increase in population P is the number of births per year minus the number of deaths per year, or $P = b - d$. Write an expression that can be used to model the population increase in the U.S. from 1990 to 2002 in function notation.

47. Assume that births and deaths continue at the same rates. Estimate the net increase in population in 2015.

SHOPPING **For Exercises 48–50, use the following information.**
Liluye wants to buy a pair of inline skates that are on sale for 30% off the original price of \$149. The sales tax is 5.75%.

48. Express the price of the inline skates after the discount and the price of the inline skates after the sales tax using function notation. Let x represent the price of the inline skates, $p(x)$ represent the price after the 30% discount, and $s(x)$ represent the price after the sales tax.

49. Which composition of functions represents the price of the inline skates, $p[s(x)]$ or $s[p(x)]$? Explain your reasoning.

50. How much will Liluye pay for the inline skates?

EXTRA PRACTICE
See pages 905 and 932.
MathOnline
Self-Check Quiz at algebra2.com

51. **FINANCE** Regina pays \$50 each month on a credit card that charges 1.6% interest monthly. She has a balance of \$700. The balance at the beginning of the nth month is given by $f(n) = f(n - 1) + 0.016 f(n - 1) - 50$. Find the balance at the beginning of the first five months. No additional charges are made on the card. (*Hint*: $f(1) = 700$)

H.O.T. Problems

52. OPEN ENDED Write a set of ordered pairs for functions f and g, given that $f \circ g = \{(4, 3), (-1, 9), (-2, 7)\}$.

53. FIND THE ERROR Danette and Marquan are trying to find $[g \circ f](3)$ for $f(x) = x^2 + 4x + 5$ and $g(x) = x - 7$. Who is correct? Explain your reasoning.

Danette

$[g \cdot f](3) = g((3)\,2 + 4(3) + 5)$
$= g(26)$
$= 26 - 7$
$= 19$

Marquan

$[g \cdot f](3) = f(3 - 7)$
$= f(-4)$
$= (-4)^2 + 4(-4) + 5$
$= 5$

54. CHALLENGE If $f(0) = 4$ and $f(x + 1) = 3f(x) - 2$, find $f(4)$.

55. *Writing in Math* Refer to the information on page 384 to explain how combining functions can be important to business. Describe how to write a new function that represents the profit, using the revenue and cost functions. What are the benefits of combining two functions into one function?

STANDARDIZED TEST PRACTICE

56. ACT/SAT What is the value of $f(g(6))$ if $f(x) = 2x + 4$ and $g(x) = x^2 + 5$?

A 38

B 43

C 86

D 261

57. REVIEW If $g(x) = x^2 + 9x + 21$ and $h(x) = 2(x + 5)^2$, which is an equivalent form of $h(x) - g(x)$?

F $-x^2 - 11x - 29$

G $x^2 + 11x + 29$

H $x + 4$

J $x^2 + 7x + 11$

Spiral Review

List all of the possible rational zeros of each function. (Lesson 6-9)

58. $r(x) = x^2 - 6x + 8$ **59.** $f(x) = 4x^3 - 2x^2 + 6$ **60.** $g(x) = 9x^2 - 1$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function. (Lesson 6-8)

61. $f(x) = 7x^4 + 3x^3 - 2x^2 - x + 1$ **62.** $g(x) = 2x^4 - x^3 - 3x + 7$

63. CHEMISTRY The mass of a proton is about 1.67×10^{-27} kilogram. The mass of an electron is about 9.11×10^{-31} kilogram. About how many times as massive as an electron is a proton? (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation or formula for the specified variable.** (Lesson 1-3)

64. $2x - 3y = 6$, for x **65.** $4x^2 - 5xy + 2 = 3$, for y **66.** $3x + 7xy = -2$, for x

67. $I = prt$, for t **68.** $C = \frac{5}{9}(F - 32)$, for F **69.** $F = G\frac{Mm}{r^2}$, for m